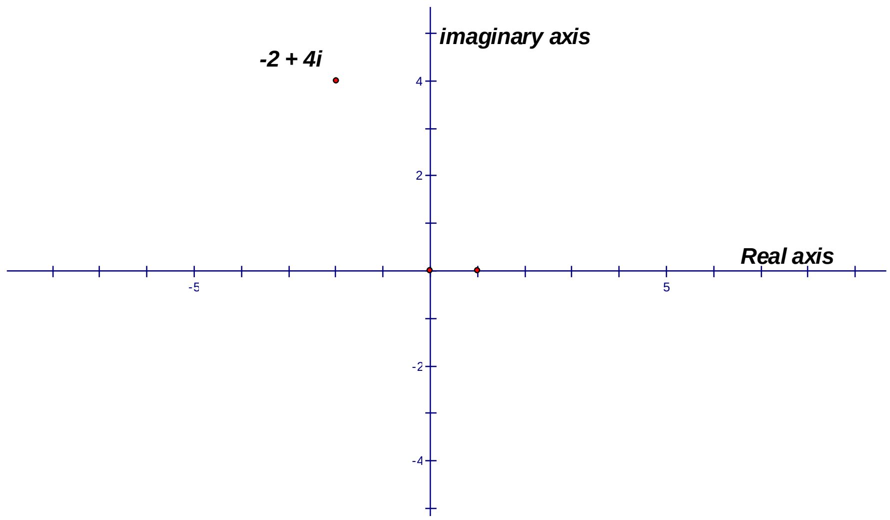
3.18 De Moivre's Theorem and *n*th Roots

<u>The Complex Plane</u>: every complex number can be associated with a point in the complex plane. A complex number in the form $\boldsymbol{z = a + bi}$ can be graphed as point (a, b) with the real number a being place along the horizontal or **real axis** and the imaginary numbers along the vertical axis or **imaginary axis**.

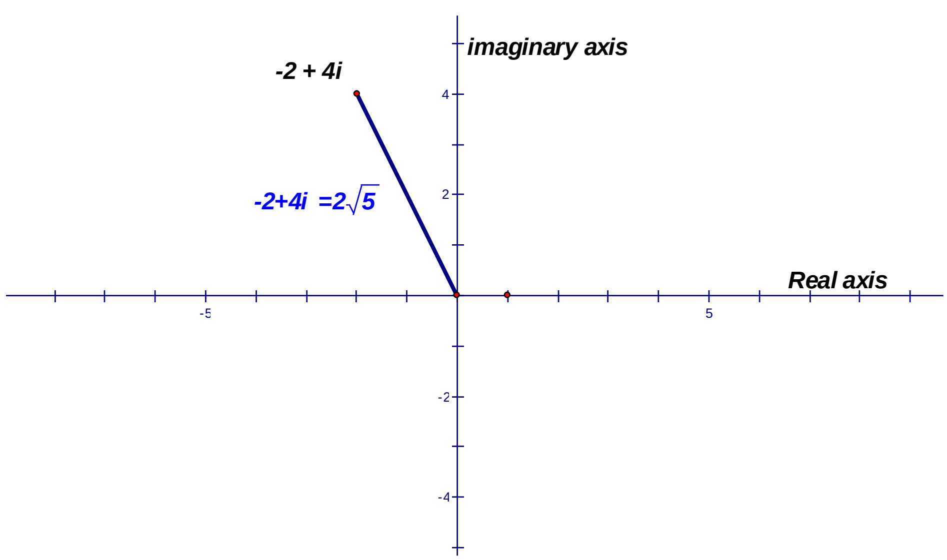
Ex.
Graph the complex number -2 + 4*i*.



<u>Absolute Value (Modulus) of a Complex Number:</u>

$$|z| = |a+bi| = \sqrt{a^2+b^2}$$

In the complex plane, $|a+bi|$ is the distance of $a+bi$ from the origin.



The trig form of the complex number $z = a + bi$ is

$$z = r\ (\cos\ \theta + i\ \sin\ \theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$. The number r is the *absolute value* or *modulus* of *z*, and θ is an **argument** of *z.*

Finding trig forms

Find the trig form with $0 \le \theta \le 2\pi$ for the complex number.

a) $1 - \sqrt{3}i$

$a = 1 \quad b = -\sqrt{3}$

$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$

reference angle $\theta = \frac{5\pi}{3}$

so, $1 - \sqrt{3}i = 2\cos\frac{5\pi}{3} + 2i\sin\frac{5\pi}{3}$

b) $-3 - 4i$

$a = -3 \quad b = -4$

$r = \sqrt{(-3)^2 + (-4)^2} = 5$

reference angle $\theta \approx 4.07$

so, $-3 - 4i \approx = 5\left(\cos 4.07 + i\sin 4.07\right)$

<u>Product and Quotient of Complex Numbers</u>

$$Let\ z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad and\ \ z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2[(\cos\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad , r \ne 0$$

Multiplying Complex Numbers

Ex. Express the product of $z_1 and\ z_2$ in standard form.

$$z_1 = 25\sqrt{2}(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}) \qquad z_2 = 14(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

Solution:

$$\begin{aligned} z_1 \cdot z_2 &= 25 \bullet 14\sqrt{2}\left[\cos\left(\frac{-\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{-\pi}{4} + \frac{\pi}{3}\right)\right] \\ &= 350\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \\ &= 478.11 + 128.11i \end{aligned}$$

De Moivre's Theorem

$Let\, z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Use De Moivre's Theorem to find $\left(1 + i\sqrt{3}\right)^3$.

Solution: We need to find the argument $\left(\theta\right)$ and the modulus (r) of $\left(1 + i\sqrt{3}\right)^3$.

$\theta = \dfrac{\pi}{3} \quad r = 2$

Now using the theorem,

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z^3 = 2^3\left[\cos\left(3 \bullet \frac{\pi}{3}\right) + i\sin\left(3 \bullet \frac{\pi}{3}\right)\right]$$

$$z^3 = 8\left(\cos\pi + i\sin\pi\right) = 8\left(-1+0i\right) = -8$$

***n*th Root of a Complex Number:**

A complex number $\boldsymbol{v = a + bi}$ is an ***nth root of z*** if $\boldsymbol{v^n = z}$ **.**

If $z = 1$, then v is an **nth root of unity.**

***n*th Roots of a Complex Number**

If $z = r\,(\cos\,\theta + i\,\sin\,\theta)$, then the n distinct complex numbers

$\sqrt[n]{r}\left(\cos\dfrac{\theta+2\pi k}{n} + i\sin\dfrac{\theta+2\pi k}{n}\right)$, where $k = 0, 1, 2, \ldots n-1$, are the

***n*th roots of the complex number z.**

Finding Fourth Roots

Find the fourth roots of $z = 5\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

$n = 4$ and $k = 0, 1, 2, 3$

The fourth roots are the numbers

$\sqrt[4]{5}\left(\cos\dfrac{\frac{\pi}{3}+2\pi k}{4} + i\sin\dfrac{\frac{\pi}{3}+2\pi k}{4}\right)$ for $k = 0, 1, 2, 3$.

$$z_1 = \sqrt[4]{5}\left(\cos\left(\frac{\pi}{12}+\frac{0}{2}\right) + i\sin\left(\frac{\pi}{12}+\frac{0}{2}\right)\right)$$

$$= \sqrt[4]{5}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$z_2 = \sqrt[4]{5}\left(\cos\left(\frac{\pi}{12}+\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{12}+\frac{\pi}{2}\right)\right)$$
$$=\sqrt[4]{5}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

$$z_3 = \sqrt[4]{5}\left(\cos\left(\frac{\pi}{12}+\frac{2\pi}{2}\right)+i\sin\left(\frac{\pi}{12}+\frac{2\pi}{2}\right)\right)$$
$$=\sqrt[4]{5}\left(\cos\frac{13\pi}{12}+i\sin\frac{13\pi}{12}\right)$$

$$z_4 = \sqrt[4]{5}\left(\cos\left(\frac{\pi}{12}+\frac{3\pi}{2}\right)+i\sin\left(\frac{\pi}{12}+\frac{3\pi}{2}\right)\right)$$
$$=\sqrt[4]{5}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$$

Finding Cube Roots

Ex.

Find the cube roots of the complex number.

$$2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$n = 3$, $k = 0, 1, 2$

$$\sqrt[3]{2}\left(\cos\frac{2k\pi+\frac{\pi}{4}}{3}+i\sin\frac{2k\pi+\frac{\pi}{4}}{3}\right)$$
$$= \sqrt[3]{2}\left(\cos\frac{\pi(8k+1)}{12}+i\sin\frac{\pi(8k+1)}{12}\right)$$

…

Finding roots of unity

Ex.

Find the fourth roots of unity and graph each root in the complex plane.

Solution:

The *n*th roots of unity are:

$$\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, k=0,1,2,\ldots,n-1.$$

So for the fourth roots of unity we would have,

$$Z_1=\cos\frac{2\bullet 0\pi}{4}+i\sin\frac{2\bullet 0\pi}{4}=1+0i$$

$$Z_2=\cos\frac{2\bullet 1\pi}{4}+i\sin\frac{2\bullet 1\pi}{4}=0+1i$$

$$Z_3=\cos\frac{2\bullet 2\pi}{4}+i\sin\frac{2\bullet 2\pi}{4}=-1+0i$$

$$Z_4=\cos\frac{2\bullet 3\pi}{4}+i\sin\frac{2\bullet 3\pi}{4}=0-1i$$